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TOPOLOGY IN 4D SIMPLICIAL QUANTUM GRAVITY

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Abstract

We simulate $4d$ simplicial gravity for three topologies S^4, $S^3 \times S^1$ and $S^1 \times S^1 \times S^1 \times S^1$ and show that the free energy for these three fixed topology ensembles is the same in the thermodynamic limit $N_4 \to \infty$. We show, that the next-to-leading order corrections, at least away from the critical point, can be described by kinematic sources.

1 Introduction

It is not a priori clear, whether in a path-integral formulation of quantum gravity the sum over metrics should also run over topologies. In a theory containing topology fluctuations only those topologies contribute to the sum in the thermodynamic limit, which maximize the extensive part of the free energy. Other contributions are exponentially suppressed. If one believes that *all* topology excitations should be present in the continuum theory, the bulk volume contribution to the free energy should be independent of topology. An explicit example of such a theory is provided by two dimensional quantum gravity. Here the Einstein-Hilbert action is of purely topological nature. It can be shown explicitly that the coefficent μ_c of the leading (extensive) part of the entropy does not depend on the topology. The coefficient of the logarithmic correction is $\gamma - 3$, where γ is the so-called surface susceptibility exponent. The exponent γ, does, however, depend linearly on the genus of the surface. The linear dependence leads to the double scaling limit at which one can reduce the number of non-perturbative modes of the theory with fluctuating topology to the solutions of the Painlevé II equation. In four dimensions the situation is more complicated. At present, no classification of topologies is known. Therefore the topological part in the action is unknown. Simplicial quantum gravity allows us to sum over geometries with fixed topology. We investigate numerically three different topologies and show that in these cases, up to the leading order, the free energy does not depend on the topology. We observe a topology dependence in the next to leading order. We study those volume corrections and analyze their sources at the kinematic bounds. Some of our results have already been presented at Lattice 96 [1].

2 Definitions

The partition function of simplicial quantum gravity in the grand canonical ensemble is

$$Z(\kappa_2, \kappa_4) = \sum_{\mathcal{T}} \frac{1}{C(T)} e^{-\kappa_4 N_4 + \kappa_2 N_2}, \tag{1}$$

where the first summation is over all $4d$-simplicial manifolds $\mathcal{T}$ with fixed topology [2]. The parameter κ_4 is proportional to the cosmological constant and κ_2 is a linear combination of the inverse of the Newton constant and the cosmological constant in the naive continuum limit. The prefactor $1/C(T)$ is a remnant of the invariance group and divides out the internal symmetry factor of the triangulation. The free energy F in the

canonical ensemble is defined as:

$$e^{F(\kappa_2,N_4)} = \sum_{T\in\mathcal{T}(N_4)} \frac{1}{C(T)} e^{\kappa_2 N_2(T)} \tag{2}$$

The sum runs over fixed topology $4d$ simplicial manifolds with a fixed number N_4 of 4-simplices. In the large volume limit $N_4 \to \infty$ the free energy is assumed to have the form :

$$F(\kappa_2, N_4) = N_4 f(\kappa_2) + \delta(\kappa_2, N_4), \tag{3}$$

where the function δ is a finite size correction, *ie* for any κ_2

$$\lim_{N_4\to\infty} \frac{\delta(\kappa_2, N_4)}{N_4} = 0. \tag{4}$$

To recover some basic properties of the theory it is convenient to study the derivatives of the free energy. The action density

$$r(\kappa_2, N_4) = \frac{1}{N_4}\frac{\partial F}{\partial \kappa_2} = \frac{\langle N_2\rangle}{N_4} \tag{5}$$

is normalized such that it becomes an intensive quantity in the thermodynamic limit $N_4 \to \infty$. It is related to the average bare Regge curvature R_{av} by $r = R_{av} + \alpha$ where $\alpha = \frac{10}{2\pi}\arccos(1/4) \approx 2.09784$. The average is taken in the canonical ensemble (2).

The derivative with respect to N_4

$$\tilde{\kappa}_4(\kappa_2, N_4) = \frac{\partial F}{\partial N_4}. \tag{6}$$

will be called the critical value of the parameter κ_4. Our definition differs slightly from the one proposed in [3]. This quantity is a measure of finite size dependence of the free energy, *ie* how fast it approaches the thermodynamic limit $\tilde{\kappa}_4 \to \tilde{\kappa}_4(\kappa_2)$ for $N_4 \to \infty$. In the thermodynamic limit $N_4 \to \infty$ the value $\tilde{\kappa}_4(\kappa_2)$ defines a critical line of the model corresponding to the radius of convergence of the series (2).

Taking the second derivative, we see that $\tilde{\kappa}_4$ is related to the action density r in the following way :

$$\frac{\partial \tilde{\kappa}_4}{\partial \kappa_2} = r + N_4 \frac{\partial r}{\partial N_4} \tag{7}$$

It is important to note that in the large N_4 limit the second term on the right hand side of (7) goes to zero, and the critical parameter $\tilde{\kappa}_4$ becomes an integral of r. Thus, if r is independent of topology in the thermodynamic limit, so is $\tilde{\kappa}_4$, unless the integration constant depends on the topology. To fix the integration constant, one just has to measure $\tilde{\kappa}_4$ for one particular value of κ_2.

3 Methods

The average action density (5) can easily be measured in canonical simulations. The quantity $\tilde{\kappa}_4$ (6) requires non-conventional methods. Some of them, like those based on the sum rules [4] or an analysis of the baby universe distributions [5], permit to directly extract the next-to-leading volume corrections. These methods are unfortunately limited to the elongated phase. Following [3], we adopt here a more general method based on multi canonical simulations which works equally well in the entire range of the coupling κ_2. To learn about how the free energy depends on N_4 one lets the volume fluctuate in the external potential $U(N_4)$. Measuring the resulting N_4 distribution and combining it with the known form of U one gets the dependence of the free energy on N_4. The freedom one has in choosing of the potential U should be used to minimize the error for the quantity one wants to measure. In this particular case one wants to measure $\partial F/\partial N_4$ for the function F, which is expected to smoothly approach a function $N_4 f(\kappa_2)$ linear in N_4 for large N_4 as given in eq. (3), (4). A Gaussian term controlling fluctuations around a fixed volume V_4 is well suited for this problem [3] :

$$U = -\kappa_4 N_4 + \frac{\gamma}{2}(N_4 - V_4)^2 \tag{8}$$

but other terms can be used as well [2]. The multicanonical partition function for this potential reads :

$$Z = \sum_T e^{\kappa_2 N_2 + U(N_4)} = \sum_{N_4} e^{F(\kappa_2, N_4) - \kappa_4 N_4 + \frac{\gamma}{2}(N_4 - V_4)^2} \tag{9}$$

Expanding $F(\kappa_2, N_4)$ around $N_4 = V_4$ one sees that the N_4 distribution becomes Gaussian :

$$P(x = N_4 - V_4) \sim \exp(-\frac{\Gamma}{2}(x - x_0)^2 + ...) \tag{10}$$

when κ_4 is tuned to be close to the derivative $\partial F/\partial V_4$ and γ is much larger than the second derivative $\partial^2 F/\partial V_4^2$. This means that the range of the distribution is much smaller than the typical scale for changes in F. The parameters Γ and x_0 are related to F and U :

$$\Gamma = \gamma - \partial^2 F/\partial V_4^2, \quad x_0 = (\partial F/\partial V_4 - \kappa_4)/\Gamma. \tag{11}$$

Both these quantities can be measured in the simulations, namely Γ from the width of the distribution and x_0 from the shift of the maximum from $N_4 = V_4$. From the numerical results one gets an estimator for the critical coupling at V_4 :

$$\tilde{\kappa}_4 = \partial F/\partial V_4 = \Gamma x_0 + \kappa_4. \tag{12}$$

In this formula, κ_4 is the coupling used in the simulation, from which one extracted x_0 and Γ. The estimator can be improved by minimizing x_0. This can be done recursively by setting $\kappa_4 = \tilde{\kappa}_4$ in the potential .

After performing measurements one can justify the validity of the assumptions used to write the Gaussian approximation (10). The value for γ should not be too large, because suppressing volume-fluctuations can spoil the mobility of the algorithm. On the other hand, it should not be too small either, because the average number of sweeps between two configurations with canonical volume V_4 grows with decreasing γ. In particular we checked that for $\gamma = 0.0001$, Γ computed from the width of the distribution, $\Gamma = 1/\sqrt{\sigma^2(N_4)}$, *i.e.* $\Delta N_4 = 100$, was equal to γ within the error bars, meaning that $\gamma \gg \partial^2 F/\partial^2 V_4$. Therefore the free energy changes very slowly in the range of the distribution width as needed for the approximation (10).

In this investigation we have performed simulations with $V_4 = 4000$, 8000, 16000, 32000, 64000.

Because the local moves [6] used in the Monte Carlo simulation preserve the topology t of the manifold, t is given by the topology of the starting configuration. In our simulation we choose to simulate a spherical S^4 topology and two different tori, namely $S^3 \times S^1$ and $S^1 \times S^1 \times S^1 \times S^1$.

The starting configuration for the sphere is, as usual, the $4d$-boundary of a 5-simplex. The manifold $S^3 \times S^1$ can be produced from a spherical manifold by taking away two separate 4-simplices and gluing together the boundaries b_1, b_2, which are created when removing the two 4-simplices. However, two vertices in a 4d simplicial manifold may have at most one common link. To avoid creation of a double connection in the gluing procedure, the distance between the vertices on b_1 to those on b_2 has to be at least three links. One can ensure this without going through a tedious check, by gluing cyclically together three copies of the original (spherical) manifold, *i.e.* $b_1/b'_2, b'_1/b''_2, b''_1/b_2$. The (double-) prime is used to distinguish the different copies. The boundaries b_1, b_2 are chosen such that they have no common vertex.

The $S^1 \times S^1 \times S^1 \times S^1$ manifold can be built out of the regular 3^4 square torus by dividing each elementary $4d$-cube into 4-simplices in the following way. For each cube we mark two points $p_0 = (0, 0, 0, 0)$ and $p_4 = (1, 1, 1, 1)$ lying on the opposite ends of the main diagonal and connect them by one of the shortest paths going along the edges of the cube. The shortest path goes through 4 edges, each in a different direction, and through three points, say p_1, p_2, p_3. There are 24 such paths. Each set of points p_0, p_1, p_2, p_3, p_4 forms a 4-simplex.

4 Finite Size Analysis and Results

The elongated phase of simplicial gravity is well described as an ensemble of branched polymers [7], [8], [9].

This means that in the large volume limit one can use the Ansatz

$$F(\kappa_2, N_4) = N_4 f_0(\kappa_2) + (\gamma - 3) \log N_4 + f_1(\kappa_2) \tag{13}$$

for the free energy. The corrections are expected to be of order $\mathcal{O}(1/N_4)$. The correction coefficient γ is assumed to depend only on the genus g of the underlying branched polymer structure.

Differentiating (13) with respect to κ_2 one sees that for the action density r (5)

$$r = f_0'(\kappa_2) + \frac{f_1'(\kappa_2)}{N_4} \tag{14}$$

the logarithmic corrections disappear. Therefore one should take the next corrections, namely $1/N_4$, into account. They can appear in r for purely kinematic reasons. To understand their origin consider the limit of large positive κ_2, in which only triangulations maximizing N_2 contribute to the sum (2). Such triangulations can be obtained from barycentric subdivisions of 4-simplices applied successively to a minimal starting configuration, possibly mixed with micro-canonical transformation, which do not change N_2 and N_4. By the minimal configuration we mean the minimal volume triangulation which maximizes N_2. For the barycentric subdivisions one gets the relation $N_2 = 5/2N_4 + c^0$, where the constant

$$c^0 = N_2^0 - 5/2N_4^0 \tag{15}$$

characterizes the initial minimal configuration (indicated by the index 0). The number N_2 of triangles is related to the action density $r = N_2/N_4 = 5/2 + c^0/N_4$. This means, the constant c^0 leads to $1/N_4$ corrections of the action density. The contributions to the sum (2) from triangulations built from non-minimal ones (*ie* smaller N_2) are suppressed exponentially by $\exp -\kappa_2(c^0 - c)$, where c characterizes the non-minimal start configuration with $c(\kappa_2) < c^0$.

For the sphere, the minimal configuration is the surface of a 5-simplex, and therefore c^0 is known. For the other topologies we extracted the number listed below by a numerical experiment. With the standard topology preserving Monte Carlo process we use a cooling procedure, in which we increase κ_4, to decrease N_4, and κ_2 to maximize N_2. In fact we have to increase κ_2 slowly compared to κ_4, because increasing κ_2 also increases the pseudo-critical coupling $\tilde{\kappa}_4(\kappa_2)$.

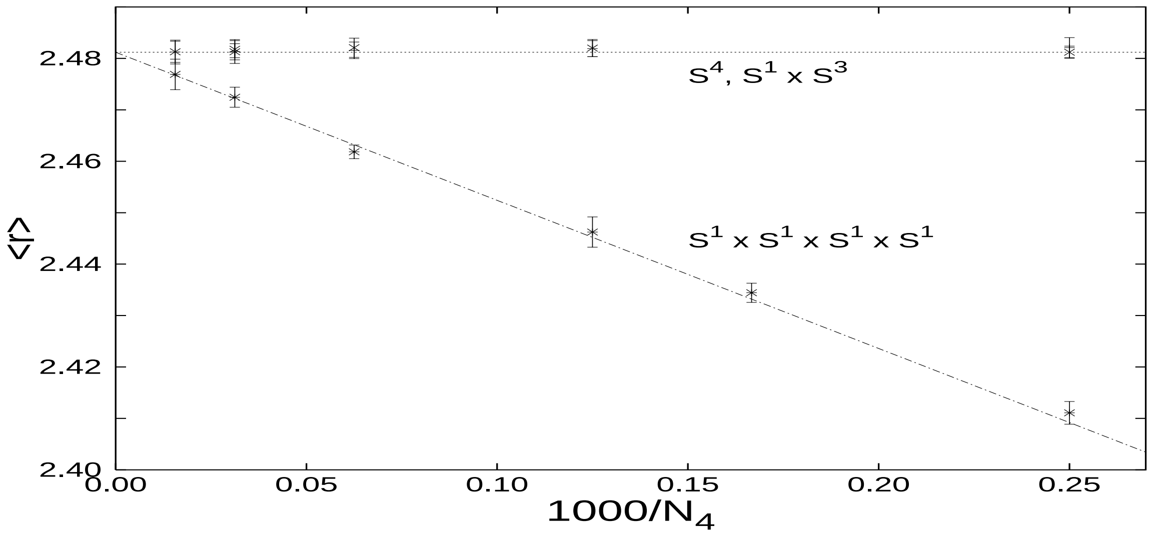

Figure 1: The volume dependence of the action density r for three different topologies in the elongated phase at $\kappa_2 = 2.0$

For the topologies studied, we found that the following configurations are minimal :

$$\begin{aligned} S^4 &: \quad N_4^0 = 6, \qquad N_2^0 = 20 \qquad \Rightarrow c^0 = 5, \\ S^3 \times S^1 &: \quad N_4^0 = 110, \quad N_2^0 = 44 \qquad \Rightarrow c^0 = 0, \\ S^1 \times S^1 \times S^1 \times S^1 &: \quad N_4^0 = 704, \quad N_2^0 = 1472 \quad \Rightarrow c^0 = 288. \end{aligned} \tag{16}$$

For the sphere S^4 the $1/N_4$ effect is very difficult to detect already for the volumes in the range of a few thousand 4-simplices and it would require extremely long runs to reduce the error bars below it. For manifolds $S^3 \times S^1$ the effect is not present at all. The corrections are, however, two orders of magnitude larger for $S^1 \times S^1 \times S^1 \times S^1$ and are measurable in the volume range used in the simulations. This estimation of the $1/N_4$ effect is exact for infinite positive κ_2 but one expects it to work although with a slowly varying coefficient $c(\kappa_2)$ in the entire elongated phase.

In fig.1 we show the action density r measured in the elongated phase for $\kappa_2 = 2.0$ against $1/N_4$. On the same figure we display the curve $r_\infty + c^0/N_4$ with $c^0 = 288$, and $r_\infty = 2.482$ which fits the data points very well. We note that that r_∞ does not depend on topology, at least for those used in the simulation.

We find, with the statistics available, no volume- or topology-dependence of $\tilde{\kappa_4}$. This is compatible with the Ansatz

$$\tilde{\kappa}_4(\kappa_2, N_4) = f_0(\kappa_2) + \frac{\gamma - 3}{N_4}, \tag{17}$$

because γ is known to be $\mathcal{O}(1)$. With the method used, the correction could only be separated from the statistical noise with an disproportionate amount of computer time. One could instead use the methods used in [5] to determine γ. We find $\tilde{\kappa}_4^{\infty} = 5.659(4)$ for $\kappa_2 = 2.0$ in the infinite volume limit.

In the crumpled phase, we use the power-law Ansatz

$$F(\kappa_2, N_4) = N_4 f_0(\kappa_2) + f_1(\kappa_2) N_4^{\delta} \tag{18}$$

Taking the derivative with respect to N_4 one gets:

$$\tilde{\kappa}_4(\kappa_2, N_4) = f_0(\kappa_2) + \delta f_1(\kappa_2) N_4^{\delta-1}. \tag{19}$$

For the action density (5) one finds

$$r = f_0'(\kappa_2) + f_1'(\kappa_2) N_4^{\delta-1} \tag{20}$$

We checked the Ansatz by fitting our numerical data for $r(N_4)$ (20):

Topology	δ	$r^{\infty} = f_0'$	$\log(f_1')$	χ^2/dof
S^4	0.5 (2)	2.039 (+0.010 / -0.013)	1.9 (1.3)	0.78
$(S^1)^4$	0.6 (2)	2.028 (+0.008 / -0.016)	1.0 (0.9)	0.87
$S^1 \times S^3$	0.5 (2)	2.038 (+0.010 / -0.021)	1.8 (1.3)	0.31

and for $\kappa_4(N_4)$ to (19):

Topology	δ	$\kappa_4^{\infty} = f_0$	$\log(\delta f_1)$	χ^2/dof
S^4	0.6 (2)	1.20 (2)	1.4 (1.3)	0.22
$(S^1)^4$	0.6 (3)	1.20 (2)	1.4 (1.5)	0.10
$S^1 \times S^3$	0.6 (2)	1.20 (2)	1.4 (1.3)	0.18

In the tables we give the logarithms of f_1' and δf_1, while these have approximately symmetric errors. One can see that the the values do not depend, within errors, on topology. In figure 2 we show the numerical data for κ_4 for the three topologies and $\delta = 0.5$.

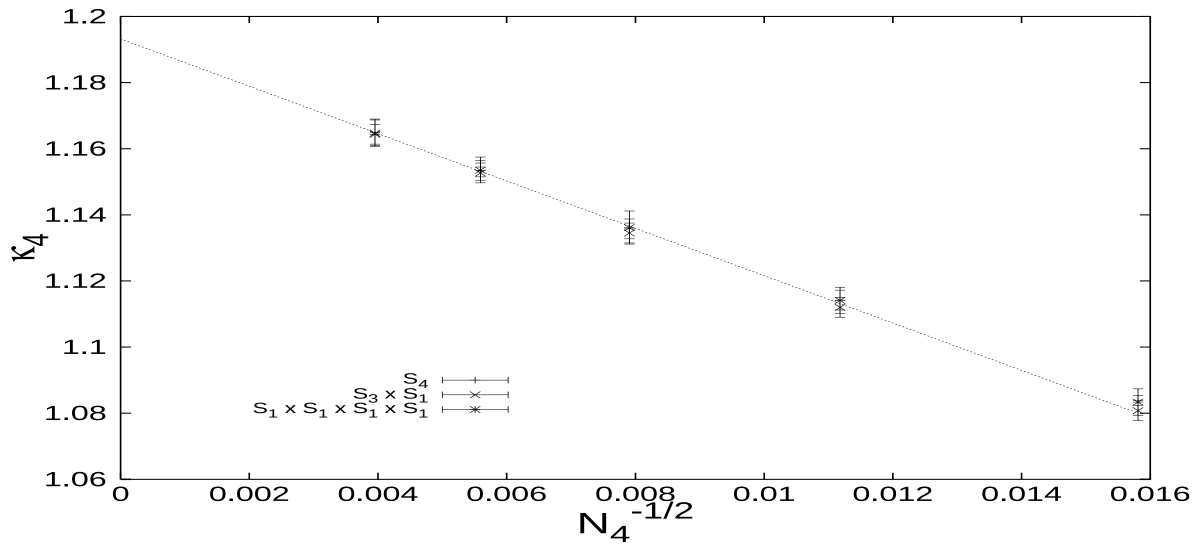

Figure 2: The volume dependence of the critical κ_4 for three different topologies in the crumpled phase at $\kappa_2 = 0.0$

The value $\delta = 0.5$ can be understood by looking at the typical configurations, that dominate the ensemble for large negative κ_2. These are configurations, which minimize the free energy, *i.e.* which have for a fixed volume the minimal number of vertices and thereby the minimal number N_2 of triangles. For the three-dimensional case such configurations were constructed in [10]. This construction can easily be extended to the four-dimensional case. One starts with a $2d$ triangulation of the sphere with t triangles. At each triangle one builds a $4d$-pancake neighborhood from q 4-simplices lying around the triangle in such a way that the links opposite to this triangle form a circle. An opposite link is defined to be a link, which does not share a common vertex with the triangle. The next step is to put neighboring pancakes together by identifying these circles with the $3d$ faces of the neighboring pancakes. Each pancake has three such faces which lie between the circle and an edge of the basic triangle. It also has three neighboring pancakes. After this step one gets an S^4 sphere with $t * q$ 4-simplices and $2 + t/2 + q$ vertices. The highest connectivity $N_4 \sim (N_0 - 2)^2/2$ is reached when $t = 2q$. The number N_2 of triangles is a linear combination of N_0 and N_4. Using equation 5 one finds $r \propto 1//sqrt(N_4)$ *ie.* $\delta = 0.5$ for these configurations. We want to note, that in [11] an argument in favor of $\delta = 0.75$ was given. This can not be excluded by our numerical data.

In [12] evidence was given, that for spherical topology the phase transition is of first

order. This was confirmed in [13], [14]. We could observe flip-flops in the action density also for the two tori under investigation. We interpret this as a hint, that the transition is of first order in these cases as well.

Finally we want to comment on the behavior of the algorithm in the crumpled phase. The typical configuration of simplicial gravity in this regime has one so-called singular link. The local volume of a singular link, *i.e.* the number of four-simplices which contain this link, diverges, when the volume of the entire configuration goes to infinity. The relaxation time, *i.e.* the number of Monte Carlo sweeps required until the singular link appeared in the configuration, was extremely long for all topologies. This effect is even more pronounced if one starts with a branched-polymer like configuration. For the $(S^1)^4$-torus and small volumes (less than 64k 4-Simplices) the situation is even worse. It was, at least for the run-length used in our numerical experiments, impossible to relax to such singular configurations. On the other hand, we know that they exist, because they could be reached by shrinking down larger configurations containing a singular link. It seems it is difficult for the algorithm to deal with two different defects, the singular link and the hole, at the same time. We will discuss this point, which is directly related to the question of practical ergodicity, more carefully in a forthcoming publication.

5 Discussion and Conclusions

In this paper we have investigated the behavior of the entropy density and the curvature for three different topologies in four dimensional simplicial gravity. We employed manifolds consisting of between 4000 and 64000 simplices. We concentrated on two values of the gravitational coupling constant κ_2, one in the crumpled and one in the branched polymer phase. We found that in both the cases the value of the entropy density and curvature in the infinite volume limit are equal for these three topologies. This gives further support to the conjecture that these limits exist, a question which was discussed in the literature some time ago [15], [11], [16].

Furthermore, it gives support to the hypothesis that all topologies contribute to a sum over topologies, like in two dimensions. The value of the entropy exponent can be compared with the estimates given in [17], based on a summation over all distributions of curvature. This estimate, which surprisingly enough is exact for the leading term in two dimensions, does, however, not directly give results in agreement with our numerical data for $\tilde{\kappa}_4$.

We further analyzed in detail the finite size effects, and compared them to estimates from simple kinematic arguments. This approach explain very well the finite size cor-

rections. It may, of course still be possible that those effects are of a more complicated nature near the transition.

Finally, as also seen in previous investigations, we observe that the algorithm has very long relaxation times in the crumpled phase. We even found that for the $(S_1)^4$ torus, the ground state seemed not to accessible using the approximately fixed N_4 algorithm, but only by passing through states with much larger N_4-values. This breakdown of practical ergodicity is still under investigation.

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